Absolute and relative measures of the demand response to changes in prices and consumer income

Classification of goods and services: ordinary, Giffen, Veblen good; substitute, complementary, independent goods; inferior, normal good

Paths of price and incomexpansion of demand

Marshallian dynamic demand function

1.4. Exercises

E1. We have:

the supply set $B = \{\boldsymbol{x} \in R_+^2 \mid x_1 \leq b_1, x_2 \leq b_2\}, b_1, b_2 > 0,$

the demand set $D(\boldsymbol{p}, I) = \{\boldsymbol{x} \in R_+^2 \mid p_1x_1 + p_2x_2 \leq I\}, p_1, p_2 > 0,$

the supply-demand set $P = B \cap D(\boldsymbol{p}, I)$ such that:

a) $P = B \cap D(\boldsymbol{p}, I) = B,$

b) $P = B \cap D(\boldsymbol{p}, I) = D(\boldsymbol{p}, I),$

c) $P \neq D(\boldsymbol{p}, I)$ or $P \neq B,$

the consumption utility functions:

a) linear $u(\boldsymbol{x}) = a_1x_1 + a_2x_2 + a_3, a_1, a_2, a_3 > 0,$

b) of Koopmans-Leontief $u(\boldsymbol{x}) = min\{a_1x_1, a_2x_2\},\ a_1, a_2, > 0.$

1. Present a geometric illustration of three demand-supply sets P in the goods space $X = R_+^2$.
2. Present illustrations of indifference curves $\{\boldsymbol{x} \in R_+^2 \mid a_1x_1 + ax_2 + a_3 = u_0 > 0\}$ for the linear utility function and Koopmans-Leontief utility function.
3. By a geometric method solve the consumption utility maximization problems with the linear utility function and the Koopmans-Leontief utility function on the sets P.
4. Can be that the consumption utility maximization problem with the linear utility function or the Koopmans-Leontief utility function has infinitely many optimal solutions on the sets of acceptable solutions P?

E2. We have:

the demand – supply set: $P = B \cap D(\boldsymbol{p}, I)$ such that:

a) $P = B \cap D(\boldsymbol{p}, I) = B,$

b) $P = B \cap D(\boldsymbol{p}, I) = D(\boldsymbol{p}, I),$

c) $P \neq D(\boldsymbol{p}, I)$ or $P \neq B.$

and the consumption utlitity maximization problem:

$u(\boldsymbol{x}) \rightarrow max,$

$\boldsymbol{x} \in P$

- if $\forall i \ z_i(\mathbf{p}(t)) = \bar{x}_i(\mathbf{p}(t)) - \bar{a}_i < 0$, then $\bar{x}_i(\mathbf{p}(t)) < \bar{a}_i \Leftrightarrow p_{i(t+1)} < p_i(t)$, which means that if the global demand for the i-th good is lower than the global supply of this good, then its price at time $t+1$ should be lower than at time t,
- it $\exists i \ z_i(\mathbf{p}(t)) = \bar{x}_i(\mathbf{p}(t)) - \bar{a}_i = 0$, then $\bar{x}_i(\mathbf{p}(t)) = \bar{a}_i \Leftrightarrow p_{i(t+1)} = p_i(t)$, which means that if on the market of the i-th good at time t the global supply and the global demand are the same, then the price of this good should not be changed at time $t+1$. In this case, when the price of the i-th good is the equilibrium price, we say that the market for all goods has a partial equilibrium with respect to the i-th good.

On the other hand, if with the proposed price system, there is an equilibrium of global supply and global demand for all goods, we say that a global equilibrium was achieved in the consumer goods market – the equilibrium defined by the Walrasian equilibrium price vector $\bar{\mathbf{p}} > \mathbf{0}$.

The main questions regarding the market described by the dynamic Arrow-Hurwicz model are:

- Is there a state of the Walrasian equilibrium on the consumer goods market?
- Is there exactly one or at least one state of the Walrasian equilibrium?
- Whether and in what period is it possible to achieve the state of the Walrasian equilibrium?

To answer these questions, one needs to determine in what way a broker sets the price of consumer goods.

Df. 2.31 A **dynamic discrete-time Arrow-Hurwicz model** is a system of difference equations of the form:

(77) $\forall i \quad p_i(t+1) = p_i(t) + \sigma_i z_i(p(t))$,

with the initial condition:

(78) $\forall i \quad p_i(0) = p_i^0 > 0$,

(79) $t = 0,1,2,...$

where: $\sigma_i > 0$ denotes a measure of the broker ‘s sensitivity to the imbalance in the i-th good's market, which for the sake of simplicity is assumed to be the same for the market of each good: $\forall i \quad \sigma_i = \sigma > 0$.

The condition (77) can be written in the equivalent form:

(80) $\forall i \quad p_i(t+1) - p_i(t) = \sigma_i z_i(p(t))$.

On the basis of (77) and (80) we can conclude that:

$z_i(p(t)) > 0 \Rightarrow p_i(t+1) - p_i(t) > 0 \Rightarrow p_i(t+1) > p_i(t)$,

$z_i(p(t)) < 0 \Rightarrow p_i(t+1) - p_i(t) < 0 \Rightarrow p_i(t+1) < p_i(t)$,

$z_i(p(t)) = 0 \Rightarrow p_i(t+1) - p_i(t) = 0 \Rightarrow p_i(t+1) = p_i(t)$.

Equivalent conditions (77) and (80) lead to a simple recursive rule for determining the prices of all goods in subsequent moments of time, which does not, however, ensure that the resulting price systems will make economic sense. We are not interested in situations where the price of any commodity is negative. Therefore, our attention should be focused only on such solutions to systems of difference equations (77) or (80), in which the vectors of consumer goods prices determined on basis of these solutions are positive: $\forall i \quad p_i(t+1) > 0$.

Df. 2.32 A **feasible price trajectory** in the dynamic discrete-time Arrow-Hurwicz model is an infinite sequence of solutions to the difference equations system (80) with an initial condition $\mathbf{p}(0) = \mathbf{p}^0 > \mathbf{0}$ such that $\forall t = 0,1,2,\ldots \boldsymbol{p}(t+1) > \mathbf{0}$.

Assuming there exists a feasible price trajectory in the dynamic discrete-time Arrow--Hurwicz model, one is interested in the conditions of existence, uniqueness and stability of the Walrasian equilibrium state.

Df. 2.33 The **Walrasian equilibrium** $\overline{\mathbf{p}} > \mathbf{0}$ is called **asymptotically globally stable** when the feasible trajectory of goods prices meets the condition:

(81) $\lim_{t \to +\infty} \mathbf{p}(t+1) \to \overline{\mathbf{p}}$.

Global stability means that any feasible trajectory of goods prices starting from any initial price system $\mathbf{p}(0) = \mathbf{p}^0 > \mathbf{0}$ after reaching the state of Walrasian equilibrium will remain in this state. It is about an asymptotic stability, because the state of Walrasian equilibrium is a target state which, if exists, can be achieved in an infinite time horizon.

Df. 2.34 A **dynamic continuous-time Arrow-Hurwicz model** is a system of differential equations of the form:

(82) $\forall i \quad \dfrac{dp_i(t)}{dt} = \sigma_i \, z_i(p(t))$,

with the initial condition:

(83) $\forall i \quad p_i(0) = p_i^0 > 0$,

(84) $t \in [0; +\infty)$,

where:

$\sigma_i > 0$ denotes a measure of the broker's sensitivity to the imbalance in the *i*-th good market, which for the sake of simplicity is assumed to be the same for the market of each good.

On the basis of (82) we can conclude that:

$$z_i(p(t)) > 0 \Rightarrow \frac{dp_i(t)}{dt} > 0 \Rightarrow p_i(t+1) > p_i(t),$$

$$z_i(p(t)) < 0 \Rightarrow \frac{dp_i(t)}{dt} < 0 \Rightarrow p_i(t+1) < p_i(t),$$

and

$$z_i(p(t)) = 0 \Rightarrow \frac{dp_i(t)}{dt} = 0 \Rightarrow p_i(t+1) = p_i(t).$$

Still a simple recursive rule for determining the prices of all commodities in subsequent moments of time, described by the condition (82), does not guarantee that the resulting price systems will make economic sense. Therefore, we should focus our attention only on such solutions to systems of differential equations (82), in which the vectors of consumer goods prices determined on basis of these solutions are positive: $\forall i \quad p_i(t + \Delta t) > 0, \quad \Delta t \to 0$.

Df. 2.35 A **feasible price trajectory** in the dynamic continuous-time Arrow-Hurwicz model is an infinite sequence of solutions to the differential equations system (82) with an initial condition $\mathbf{p}(0) = \mathbf{p}^0 > \mathbf{0}$ such that $\forall t \in [0; +\infty) \quad \mathbf{p}(t + \Delta t) > \mathbf{0}$.

Assuming that there exists a feasible price trajectory in the dynamic continuous-time Arrow-Hurwicz model, one is interested in conditions of existence, uniqueness and stability of the Walrasian equilibrium state.

Df. 2.36 The state of Walrasian equilibrium $\overline{\mathbf{p}} > 0$ is called asymptotically globally stable when the feasible trajectory of prices meets the condition:

(85) $$\lim_{\substack{t \to +\infty \\ \Delta t \to 0}} \mathbf{p}(t + \Delta t) \to \overline{\mathbf{p}}.$$

Global stability means that any feasible trajectory of goods prices starting from any initial price system $\mathbf{p}(0) = \mathbf{p}^0 > \mathbf{0}$ after reaching the state of Walrasian equilibrium will remain in this state. It is about an asymptotic stability, because the state of Walrasian equilibrium is a target state which, if exists, can be achieved in an infinite time horizon.

Example 2.3

Two traders come to the market with bundles of goods: $\mathbf{a}^1 = (10,20)$, $\mathbf{a}^2 = (20,10)$. The utility functions of traders are: $u^1(x_{11}, x_{12}) = x_{11}^{1/4} x_{12}^{1/4}$, $u^2(x_{21}, x_{22}) = x_{21}^{1/4} x_{22}^{1/3}$. We know from Example 2.2 that in the static Arrow-Hurwicz model for the given initial allocation and given utility functions, the excess demand function takes the form:

$$\mathbf{z}(\mathbf{p}) = \left(15\frac{p_2}{p_1} - 15, 15\frac{p_1}{p_2} - 15\right),$$

and the price vector of the Walrasian equilibrium has the structure:

$$\bar{\mathbf{p}} = \lambda(1, 1), \quad \lambda > 0.$$

Let us first consider the dynamic discrete-time Arrow-Hurwicz model. The broker announces the initial prices $\mathbf{p}(0) = (2, 4)$.

1. Find the trajectories of the price vector satisfying the system of equations of the dynamic discrete-time Arrow-Hurwicz with the proportionality coefficient σ equal to 0.25, 0.35 and 1.25. Calculate price ratios $\frac{p_2(t)}{p_1(t)}$ and compare them with the equilibrium price ratio $\frac{\bar{p}_2}{\bar{p}_1}$.
2. Determine which of trajectories determined in point 1 are feasible.
3. Determine if and when the price structure stabilizes around the equilibrium structure and whether it reaches this structure in the time horizon $T = 15$.
4. Present graphs of the price trajectories in the state space.
5. Present graphs of the price trajectories as functions of time.

Ad 1

The price trajectories of the first and second goods, respectively, are determined from the formulas:

$$p_1(t+1) - p_1(t) = \sigma\left(15\frac{p_2(t)}{p_1(t)} - 15\right),$$

$$p_2(t+1) - p_2(t) = \sigma\left(15\frac{p_1(t)}{p_2(t)} - 15\right).$$

Table 2.1. Price trajectories when $\sigma = 0.25$

t	p_1	p_2	$\frac{p_2(t)}{p_1(t)}$	$\left\lvert\frac{\bar{p}_2}{\bar{p}_1} - \frac{p_2(t)}{p_1(t)}\right\rvert$
0	2	4	2	1
1	2.75	3.625	1.318182	0.318181818
2	2.988636	3.443966	1.152353	0.152353481
3	3.102901	3.344807	1.077961	0.077961226
4	3.161372	3.290565	1.040866	0.040866096
5	3.192022	3.261119	1.021647	0.02164683
6	3.208257	3.245228	1.011524	0.011523673
7	3.2169	3.236684	1.00615	0.006149976
8	3.221512	3.232099	1.003286	0.003286371
9	3.223977	3.229643	1.001757	0.001757333
10	3.225295	3.228327	1.00094	0.000940043
11	3.226	3.227623	1.000503	0.000502949

12	3.226377	3.227246	1.000269	0.000269119
13	3.226579	3.227044	1.000144	0.000144009
14	3.226687	3.226936	1.000077	7.70629E-05
15	3.226745	3.226878	1.000041	4.12391E-05
16	3.226776	3.226847	1.000022	2.20687E-05
17	3.226792	3.226831	1.000012	1.18099E-05
18	3.226801	3.226822	1.000006	6.31997E-06
19	3.226806	3.226817	1.000003	3.3821E-06
20	3.226809	3.226814	1.000002	1.80991E-06

Table 2.2. Price trajectories when σ = 0.35

t	p_1	p_2	$\frac{p_2(t)}{p_1(t)}$	$\left\lvert \frac{\bar{p}_2}{\bar{p}_1} - \frac{p_2(t)}{p_1(t)} \right\rvert$
0	2	4	2	1
1	7.25	1.375	0.189655	0.810344828
2	2.99569	23.80682	7.947024	6.947024199
3	39.46757	19.21744	0.486917	0.513082654
4	36.77388	24.74956	0.67302	0.326979955
5	35.05724	27.30022	0.778733	0.221267248
6	33.89558	28.79194	0.84943	0.150569556
7	33.10509	29.72255	0.897824	0.102175844
8	32.56867	30.32002	0.930957	0.069043271
9	32.20619	30.70938	0.953524	0.04647586
10	31.9622	30.96527	0.968809	0.031190638
11	31.79845	31.1343	0.979114	0.020886168
12	31.68879	31.24629	0.986036	0.013964048
13	31.61548	31.32064	0.990674	0.009325909
14	31.56652	31.37006	0.993776	0.006223687
15	31.53385	31.40294	0.995849	0.004151311
16	31.51205	31.42482	0.997232	0.002768058
17	31.49752	31.4394	0.998155	0.001845296
18	31.48783	31.4491	0.99877	0.001229958
19	31.48137	31.45557	0.99918	0.000819729
20	31.47707	31.45988	0.999454	0.000546287
21	31.4742	31.46274	0.999636	0.000364042

1. Formulate the monopolist's profix maximization problem and present its geometric illustration.
2. Find the optimal production quantity.
3. Find the optimal product price.
4. Find the maximum profit for the monopolist.
5. Determine the relationship between the optimal price of a product and the price elasticity of product demand.
6. Analyze the sensitivity of the optimal production quantity, maximum profit and optimal price to changes in the parameters of the demand function and the total cost function.

Note: Perform tasks 1 – 6 separately for the two different total cost functions.

E2. There is a monopoly company that produces one product and allocates it to two markets. There are given:

the exogenous functions of demand for a homogeneous product on both markets:

$y_i^d(p_i(y_i^s)) = -a_i p_i(y_i^s) + b_i, i = 1,2.$

the inverse functions of demand on both markets:

$p_i(y_i^s) = \frac{b_i - y_i^d}{a_i}$, and $p_i(y_i^s) = e_i - f_i(y_i^d),\ e_i = \frac{b_i}{a_i}, f_i = \frac{1}{a_i}, , i = 1,2$

a total quantity of the product produced by the monopolist for both markets:

$y = \xi(y_1, y_2) = y_1 + y_2$

the total production cost:

$k^c(y) = k^z(y) + k^s(y) = cy + d = c(y_1 + y_2) + d, \quad c, d > 0$

Everywhere else we assume that: $\forall i = 1,2 \quad y_i^d = y_i^d(p_i) = y_i^s(p_i) = y_i$, which means that the level of production in each market adjusts to the demand that consumers report on each of them. However, it is not possible to resell the product on a different market than the one for which it is intended.

Knowing that the monopolist's goal is to maximize profit:

1. Formulate the profit maximization problems:
   a. on the first market,
   b. on the second market,
   c. on both markets.
2. Determine the optimal level of production:
   a. on the first market,
   b. on the second market,
   c. on both markets.

3. Determine the optimal profit of the monopolist:
   a. on the market 1,
   b. on market 2,
   c. on both markets.
4. Determine the optimal product price level
   a. on market 1,
   b. on the market 2.
5. Perform a sensitivity analysis:
   a. separately for each market: optimal production level and optimal product price which ensure the maximum profit for the monopoly,
   b. for the optimal total supply of the product to both markets, with respect to changes in values of the parameters of demandfunction and the production cost function.
6. Determine the relationship between the price elasticity of demand and the optimal price of the product:
   a. on the market 1,
   b. on the market 2.
7. Determine when:
   a. the optimal price on market 1 will be higher than the optimal price on market 2,
   b. the optimal price in market 2 will be higher than the optimal price in market 1,
   c. two optimal prices will be equal to each other.

E3. There is some company which has the monopoly for its product. The demand for this product is described by the demand function: $y^d(p) = -2p + 20$. The total production cost for the company is described by the total cost function: $k^c(y) = 2y + 2$.

a. What is the optimal product supply for this monopoly?
b. What will be the level of price set by the monopolist?
c. What will be the level of profit achieved by the monopolist?

Chapter 4. Quantity and price competition in duopoly

In Chapter 3 we have dealt with rational decisions of the monopoly in a single-product market with an exogenous function of demand for a product. Let us now consider a different market structure which, due to the analysis framework we adopt, is a duopoly.

Df. 4.1 If on the market of a certain product (good or service) we have two producers, each of them having an impact on its price and output level and seeking to maximize its own profit (there is no cartel collusion, they do not maximize the joint profit), this market structure is called a **duopoly**.

Df. 4.2 If on the market of a certain product (good or service) we have n producers ($n > 2$), each of them having an impact on its price and output level and seeking to maximize its own profit (there is no cartel collusion aimed at maximizing the joint profit), then such a market structure is called an **oligopoly**.

Among the duopoly and oligopoly models one should distinguish quantity competition models (on quantity of a product) and price competition models (on price of a product).

In quantity competition models, which include Cournot and Stackelberg duopoly and oligopoly models, one assumes that producers produce homogeneous (undifferentiated) products (good or service). In that case they have to set the same price for the product. Thus, they cannot compete on the price of the product they manufacture, but can compete with each other on output levels.

In price competition models, which include the duopoly and Bertrand oligopoly models, one assumes that producers produce substitute (differentiated) products. In that case, they may set different prices for the products they manufacture. Thus, they can compete on the prices of the products they manufacture.

Due to the convention adopted in the book, we will not analyze the oligopoly models. We simply stress that they are simple generalizations of duopoly models (Cournot, Stackelberg and Bertrand) for cases where the number of producers on the market is $n > 2$.

4.1. Cournot duopoly model

4.1.1. Static approach

(C1) There are two producers ($i = 1,2$) on the market of homogeneous (undifferentiated) product. Their production total cost functions are as follows:

$$(1) \qquad \forall i=1,2 \quad k_i^c(y_i)=k_i^z(y_i)+k_i^S(y_i)=c_i y_i+d_i, \quad c_i,d_i>0$$

being the sum of variable cost functions:

$$(2) \qquad \forall i=1,2 \quad k_i^z(y_i)=c_i y_i, \quad c_i>0,$$

and the fixed cost:

$$(3) \qquad \forall i=1,2 \quad k_i^S(y_i)=d_i>0.$$

Since the total cost functions are linear functions of the production quantity, then:

$$(4) \qquad \forall i=1,2 \quad \frac{\mathrm{d}k_i^c(y_i)}{\mathrm{d}y_i}=\frac{\mathrm{d}k_i^z(y_i)}{\mathrm{d}y_i}=c_i>0,$$

the total and variable costs of the i-th producer are increasing in the output level.

(C2) The function of demand reported for a product by consumers, depending on its price set by producers, is as follows:

$$(5) \qquad y^d(p)=-ap+b, \quad a,b>0,$$

where:

a – a measure of consumers' reaction to a change in the price of a product,

b – a measure of market capacity.

Since the demand function has to be non-negative, then:

$$(6) \qquad p\in\left[0,\frac{b}{a}\right].$$

(C3) The total quantity of production by both producers matches the demand that consumers report at the given price of the product:

$$(7) \qquad y_1+y_2=y^d(p)=-ap+b, \quad a,b>0.$$

(C4) The first producer wants to determine such an output level that, taking the output level set by the second producer as given, guarantees the maximum profit for him/her:

$$(8) \qquad \Pi_1(y_1)\Big|_{y_2=\text{const.}\geq 0}\to\max$$
$$y_1\geq 0.$$

(C5) The second producer wants to determine such an output level that, taking the output level of the first producer as given, guarantees the maximum profit for him/her:

$$(9) \qquad \Pi_2(y_2)\Big|_{y_1=\text{const.}\geq 0}\to\max$$
$$y_2\geq 0.$$

The profit function of the i-th producer can be expressed as the difference between his/her function of the revenue from the sale of the product and the function of the total cost of production:

(10) $\forall i = 1,2 \quad \Pi_i(y_i) = p(y)y_i - c_i y_i - d_i = (p(y) - c)y_i - d_i,$

Substituting the inverse demand function $p(y) = \frac{b-y}{a} = \alpha - \beta(y_1 + y_2)$ where: $\alpha = \frac{b}{a}, \beta = \frac{1}{a},$ into the system of equations (10) we will obtain the profit functions of both producers as functions of their output levels:

– for the first producer:

(11) $\Pi_1(y_1, y_2) = [\alpha - \beta(y_1 + y_2)]y_1 - c_1 y_1 - d_1 = [\alpha - c_1]y_1 - \beta y_1^2 - \beta y_1 y_2 - d_1,$

– for the second producer:

(12) $\Pi_2(y_1, y_2) = [\alpha - \beta(y_1 + y_2)]y_2 - c_2 y_2 - d_2 = [\alpha - c_2]y_2 - \beta y_2^2 - \beta y_1 y_2 - d_2.$

When the output level set by the second producer is taken as given, thus treated as a parameter, the necessary and sufficient conditions for the profit maximization problem of the first producer are following[15]:

(13) $\left.\frac{\partial \Pi_1(y_1, y_2)}{\partial y_1}\right|_{y_1 = \bar{y}_1,\, y_2 = \text{const.} \geq 0} = 0$ – necessary condition,

(14) $\left.\frac{\partial^2 \Pi_1(y_1, y_2)}{\partial y_1^2}\right|_{y_1 = \bar{y}_1,\, y_2 = \text{const.} \geq 0} < 0$ – sufficient condition.

When the output level set by the first producer is taken as given, thus treated as a parameter, the necessary and sufficient conditions for the profit maximization problem of the second producer are following:

(15) $\left.\frac{\partial \Pi_2(y_1, y_2)}{\partial y_2}\right|_{y_2 = \bar{y}_2,\, y_1 = \text{const.} \geq 0} = 0$ – necessary condition,

(16) $\left.\frac{\partial^2 \Pi_2(y_1, y_2)}{\partial y_2^2}\right|_{y_2 = \bar{y}_2,\, y_1 = \text{const.} \geq 0} < 0$ – sufficient condition,

Determining the necessary and sufficient conditions for the profit function described by equations (11)–(12) we get:

– for the first enterprise:

[15] The profit function of the first (second) producer is a one variable function when the supply of a product by the second (first) producer is set. In conditions (13)–(16), we do use notions appropriate for first and second order partial derivatives, but the necessary and sufficient conditions of the optimum existence refer de facto to one variable functions.