PREFACE

This textbook contains materials for several courses that are taught in the Master Programme in Financial Engineering in the Poznań University of Economics and Business. The realm of financial engineering (or quantitative finance) is very broad and we have limited space, so we had to select some materials that we found the most important. As such topics we recognize investments, risk assessment and pricing. The chapters of this book are connected with these areas.

Chapter one covers methods used in the portfolio analysis. Starting from the classic Sharpe's Capital Asset Pricing Model, it moves to more refined and modern multifactor models of investment returns. The chapter contains real-life examples and cases from Polish and worldwide markets.

The second chapter describes basic methods of risk measurement in finance. It provides the definition of a risk and describes sources and types of risks one can encounter in financial institutions. It also provides the main measures of risk–both market risk as well as credit risk.

The third chapter provides an introduction to methods used in the derivative instruments pricing. The techniques of building formal models of financial markets are presented here. In the chapter basic notions connected with mathematical modelling in finance (such as arbitrage, risk-neutral measures and martingale pricing) are described. Due to lack of space only the discrete models (i.e. models with finite time horizon and sample space) are presented here.

Chapter four is devoted to corporate finance. It describes the main types of securities offered by companies to finance their economic activities. The main aims of issuing securities and methods of offering them are presented. The chapter contains also a description of practices from the Polish market and contains examples concerning this market.

The fifth chapter deals with modelling the term structure of interest rates. Starting from the basic concepts connected with time value of money, it introduces and describes various types of interest rates. The methods of estimating terms structure of interest rates from bonds' prices are presented here. The chapter ends with the description of the main models of term structure of interest rates that are used by central banks worldwide.

Chapter six provides broader view on the methods presented in the previous chapter and is related to the market practice. It contains information about the usage of term structure of interest rates in pricing swap instruments. The main swap instruments in the Polish financial markets are presented here. In the chapter it is shown that after the crisis of 2007-2009 more advanced methods, assuming the existence of many yield curves, are needed in practice.

The seventh chapter is devoted to hedge funds and their investment strategies. It presents an overview of the history of hedge funds and the reasons for their existence. Then it describes investment strategies used by such funds– in particular, strategies that make use of derivative instruments. The chapter ends with the examples of such strategies.

All the authors hope that this textbook will be helpful for the students of the financial engineering programme, but also for all who want to develop their knowledge in finance and, in particular, in quantitative methods used in finance.

CHAPTER 1
MULTIFACTOR MODELS: PORTFOLIO THEORY

The capital asset pricing model (CAPM), is one of the most elegant and appealing models in finance. This theory was independently developed by Treynor (1961), Sharpe (1964), Lintner (1965) and Mossin (1966) on the fundamentals of Markowitz's efficient portfolio theory (Perold, 2004). Without exaggeration the CAPM has probably been one of the most useful and frequently used financial economic theories ever developed. It has also been widely discussed and questioned. In this chapter we will focus on the application of CAPM model to the Polish capital market, which according to Modern Index Strategy indexes delivered by Morgan Stanley Capital International (MSCI) classification, belongs to the European emerging markets.[1] First, we will start with single-factor models, CAPM, and then show how the most successful extensions used in the literature are employed for this particular market. We will also examine if multifactor models provide a better explanation for the behavior of stocks returns on the Warsaw Stock Exchange.

1.1. Capital asset pricing model

Let us start with the assumptions of the single and multifactor models. As the capital market theory is based on Markowitz's portfolio theory, the assumptions are nearly identical to those used in Markowitz's approach and may be summarized as follows (Reilly & Brown, 2002):

1. All investors are directed by the risk-return distribution. They want to obtain the market portfolio or any other portfolio that is on the efficient frontier. The choice of the exact portfolio depends on the personal utility function.
2. All investors have homogenous expectations.
3. Investors can borrow and lend money at the same risk-free rate.

[1] https://www.msci.com/market-classification.

4. All investors have the same investment horizon – the models are developed for a single hypothetical period.
5. There are no transaction costs, no taxes and no inflation. The investments are perfectly divisible. The capital markets are in equilibrium.

We start with the capital asset pricing model which represents the relationship between the risk and the expected rate of return. In the CAPM the expected excess return on any single risky asset, that is the difference between the expected return on the asset and the risk-free rate return, is proportional to the expected excess return on the market portfolio (Alexander, 2008). The excess return of a given asset depends on the excess market return with a special coefficient β:

$$E\left(R_i\right)-r_F=\beta_i\left(E\left(R_M\right)-r_F\right). \tag{1}$$

Without the expectations the formula is as follows:

$$R_i=r_F+\beta_i\left(R_M-r_F\right), \tag{2}$$

where $\beta_i=\dfrac{\operatorname{cov}(R_i, R_M)}{\operatorname{var}(R_M)}=\dfrac{\sigma_i \sigma_M \rho_{iM}}{\sigma_M^2}$,

r_F – is a risk-free rate,
R_i – is return from asset i,
R_M – is return from market portfolio,
M, σ_i, σ_M – stand for standard deviations of returns of asset i and the market portfolio,
ρ_{iM} – describes the correlation coefficient between returns.

On the basis of this return-generating model one easily obtain the required (proper) return from the investment in asset i given the actual market conditions. One can compare the required rate of return with the estimated rate of return to asses if the asset is overvalued, undervalued or properly valued. The difference between expected returns (1) and realized returns (2) is due to errors that appear when the expectations are related to the realized values:

$$e_i=R_i-r_F-\beta_i\left(R_M-r_F\right). \tag{3}$$

The beta coefficient in this approach represents assets' sensitivity to the market portfolio changes and as such it is perceived as a measure of a systematic risk. As it relates the covariance to the variance of market portfolio, it is also a standardized measure and thus can be compared across the stocks listed on a given market. If beta is higher than 1, then the asset is said to be aggressive, it means it both grows and decreases faster than the market portfolio. We may also say that the asset has a higher standardized systematic risk than the market and thus it is more volatile than the overall market portfolio. If the beta is equal to one, then the portfolio behaves as a market portfolio. Beta in the interval (0,1) means that the asset grows more slowly than the market and decreases also slowly; such an asset has lower volatility than the market. The most interesting is a negative beta case: it happens when the asset returns are changing in different direction than the market portfolio returns. There are many approaches to calculate beta in the real world and we discuss them in subchapter 1.2.2.

The Security Market Line (SML) is derived from the CAPM (1) model, where the expected return $E(R_i)$ depends on the beta coefficient, β. The SML is often used for valuation and allows to examine if a given asset is undervalued, overvalued or properly valued. Based on the actual characteristics of the market, the return of the market portfolio, beta of an instrument and the risk-free rate, one is able to examine if the asset returns fit the SML. In case the asset return is above the SML, this instrument is undervalued, whereas if the return is below the SML, it is overvalued. However, in the equilibrium, all single assets should "lie" on the SML.

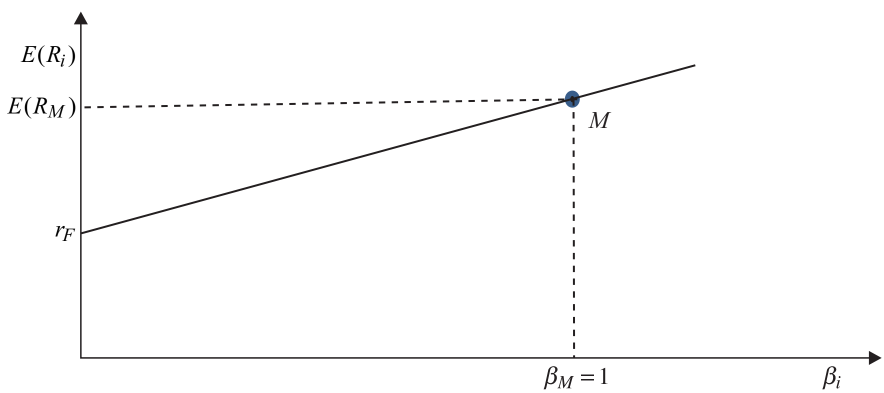


Figure 1. The Securities Market Line

An investor considers not only the return, but also risk of the stock. In the CAPM approach risk is measured as the variance of the returns and based on Eq. (2) is expressed in the following way:

$$\operatorname{var}(R_i) = \operatorname{var}\left(r_F + \beta_i (R_M - r_F) + e_i\right). \tag{4}$$

Taking into account that r_F and β are fixed:

$$\operatorname{var}(R_i) = \beta_i^2 \operatorname{var}(R_M - r_F) + \operatorname{var}(e_i) + 2\operatorname{cov}\left(\beta_i (R_M - r_F)(e_i)\right). \tag{5}$$

As we assume also that market portfolio return should be not correlated with the error term, we obtain:

$$\operatorname{var}(R_i) = \beta_i^2 \operatorname{var}(R_M) + \operatorname{var}(e_i). \tag{6}$$

Thus the total variance is decomposed into a systematic variance (measured by the beta coefficient) and a non-systematic (idiosyncratic) variance. The systematic risk is due to the market as a whole and is non-diversifiable, while the non-systematic risk can be diversified by increasing the number of the assets in the portfolio.

Since Ang, Hodrick, Xing and Zhang (2006), there is an ongoing discussion about the idiosyncratic volatility *IVOL* puzzle. This phenomena appears when the performance of stocks with a low idiosyncratic risk outperforms that of stocks with a high idiosyncratic risk. Stambaugh, Yu and Yuan (2015) find that the relationship between idiosyncratic volatility and return is negative only among overpriced stocks, while among underpriced stocks this relationship is positive. However, Zaremba (2018) shows that this feature comes out of the mathematical properties of return distributions.

Problems and solutions

Problem I

Consider the following characteristics of market and stock X: the risk-free rate is 2%, the expected rate (based on fundamental value) of return from market portfolio is 6% and the risk measured by its standard deviation is 4%, while for X asset it is 5%. The correlation coefficient of asset X' returns with the market portfolio returns is +0.2. The stock is priced at 100 EUR and is supposed to be worth 107 within a year. Calculate beta for asset X and find out if stock X is undervalued or overvalued.

Solution

The expected return for stock X is: $E(R_i) = (107 - 100)/100 = 7\%$
Beta for stock X: As $\rho_{MX} = 0.2$,

$$\beta = \frac{\operatorname{cov}(R_X, R_M)}{\operatorname{var}(R_M)} = \frac{\sigma_M \sigma_X \cdot \rho_{MX}}{\sigma_M^2} = \frac{\sigma_X \cdot \rho_{MX}}{\sigma_M} = \frac{5 \cdot 0.2}{4} = 0.25$$

(and $\operatorname{cov}(R_X, R_M) = \sigma_M \sigma_X \cdot \rho_{MX} = (4 \cdot 5)/0.2 = 100$).

The return based on the SML is:

$$E(R_X) = r_F + \beta\left(E(R_M) - r_F\right) = 2 + 0.25 \cdot (6-2) = 3\%.$$

As our expected rate of return is 7%, and this is higher than 2.5% from SML, asset X is undervalued.

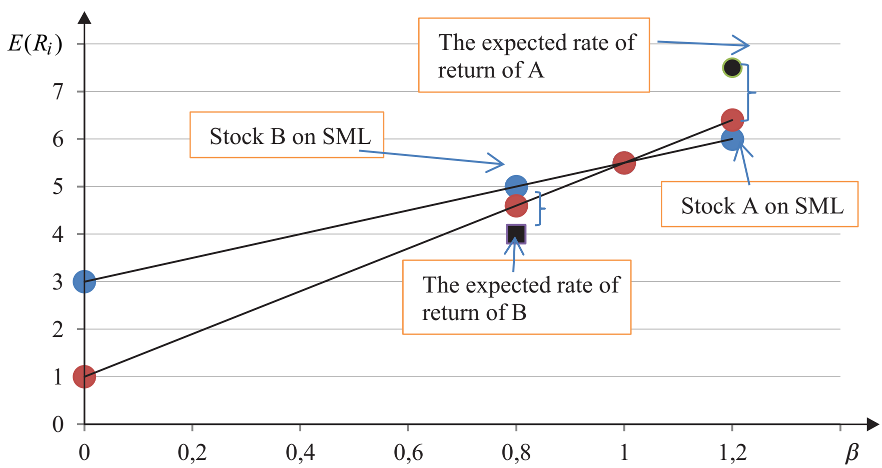
Problem II

An analyst expects a risk-free rate of 3%, a market return of 5.5% with a risk measured with standard deviation of 3%. The characteristics for stocks A and B are shown below:

Stock	beta	Standard deviation (%)	Expected rate of return (%)
A	1.2	5	7.5
B	0.8	2.5	4

1. Draw SML and find out if the stocks are fairly valued by the market (under/overvalued)?
2. Will your conclusion change if the risk-free rate decreases to 1%?

Solution



Blue bullets are for the SML line with $r_F = 3\%$, while red bullets are for the SML line with $r_F = 1\%$.

The black bullet depicts the expected rate of return of A–the difference between the black and blue (red) bullet for a given value of beta shows that asset A is undervalued. On the contrary, the black square depicts the expected rate of return of B that irrespectively of the value of risk-free rate is overvalued.

The results of calculations are presented below:

Stock	beta	Standard deviation (%)	Expected rate of return	SML valuation under $r_F = 3\%$ (%)	SML valuation under $r_F = 1\%$ (%)
A	1.2	5	7.5	**6**	**6.4**
B	0.8	2.5	4	**5**	**4.6**
M	1	3	5.5		

With respect to the second question, the conclusion is not changing as the risk-free rate moves from 3% to 1%–both the undervalued and overvalued asset are still under- or overvalued.

The SML requires that the risk-free rate is known. As it is often controversial which rate would be the best proxy for the risk-free rate and should be taken into account, a solution to this problem is to use the market model which does not contain a risk-free rate.

1.2. The characteristic line–market model

1.2.1. Model specification

The market model, also called the characteristic line of the security, for an asset i has the following specification:

$$E(R_i) = \alpha_i + \beta_i E(R_M) + e_i, \tag{7}$$

where $\alpha_i = (1-\beta_i) r_F$, α and β are parameters, and e is an error term.

The assumptions for the characteristic line usually are the following:

1. $E(e_i) = 0$
2. Variance of the error term is homoscedastic: $\mathrm{var}(e_i) = \sigma_{e_i}^2$.
3. The error terms are not correlated: $\mathrm{cov}(e_i, e_j) = 0$ for each $i \neq j$.
4. The covariance of error term with market portfolio return is zero: $\mathrm{cov}(e_i, R_M) = 0$ for each i.

Without the expectations the market model is the following:

CHAPTER 3
INTRODUCTION TO DERIVATIVE INSTRUMENTS PRICING

3.1. Pricing in one-period model

3.1.1. Stochastic model of market

We will consider finite models of markets – i.e. models with discrete time in which prices of all the available assets can take values from a finite set of numbers. In this subchapter we will study the simplest version of such a model, namely we consider only two trading dates. This is obviously very unrealistic oversimplification of real changes in stock and bond prices, but it allows to point out some important features of stochastic models of markets and develop basic relationships that hold true also in much more complicated models.

We assume that there are only two trading dates: the initial date $t=0$ and the terminal date $t=T$. We have all the information about events and prices at the initial date – it is "the present moment". However, we do not know what will happen in the future. The prices at the terminal date are modelled as random variables. To simplify, we assume that the sample space is finite. There are M possible outcomes (or **states of the world**) in the future. The sample space is thus defined as follows:

$$\Omega=\{\omega_1, \omega_2, \ldots, \omega_M\}.$$

Each state of the world could happen with some positive probability. The probability measure (called **real probability**) of the state of the world ω is given by $P(\omega)>0$. Formally the stochastic setup of the model is defined by finite probability space (Ω, F, P), where F is a σ-algebra of all subsets of Ω.

In the financial market there are $N+1$ financial assets, which are labelled form 0 to N. The prices of the asset n at the moment t is given by $S_n(t)$, where $n=0,1,\ldots,N$, $t=0,T$. We assume that all prices are non-negative (at the moment $t=0$ and at the moment $t=T$ in all possible states of the world). The prices at the moment $t=T$ can be different in different states of the world.

The prices of the first instrument, $S_0(t)$, are strictly positive. We take this instrument as a **numéraire** – the values of all the other instruments and portfolios will be measured in the units of the numéraire. Traditionally, a numéraire is assumed to be a riskless security and one uses the terms “bond” or “bank account” to describe it. In this approach, if the risk-free interest rate from the moment 0 to the final moment T equals $r > 0$, then the prices of the first instrument are given by

$$S_0(0)=1 \quad \text{and} \quad S_0(T)=1+r.$$

Generally, a numéraire does not have to be riskless. However, without losing generality we can assume that $S_0(0)=1$. Let us define the discount factor as: $\beta(t)=1/S_0(t)$. Thus $\beta(0)=1$ and $\beta(T)=\dfrac{1}{1+r}$.

The prices at the initial moment are known to the investor, but the prices at the moment T are random variables. Using vector notation we can write:

$$S(t)=\left(S_0(t), S_1(t), \ldots, S_N(t)\right)^T.$$

$S(0)$ is a $(N+1)$-dimensional vector and $S(T)$ is $(N+1)$-dimensional random variable. The prices of assets at the terminal date are random variables and depend on the state of the world. We state this dependence explicitly expressing the prices in the state of the world ω by the vector $S(T,\omega)=\left(S_0(t,\omega), S_1(t,\omega), \ldots, S_N(t,\omega)\right)^T$.

Define also $\tilde{S}(t)=\left(\tilde{S}_0(t), \tilde{S}_1(t), \ldots, \tilde{S}_N(t)\right)^T=\left(1, \beta(t)S_1(t), \ldots, \beta(t)S_N(t)\right)^T$ as the vector of **discounted** assets' prices, i.e. prices measured in the units of numéraire. Again, $\tilde{S}(0)$ is $(N+1)$-dimensional vector and $\tilde{S}(T)$ is a vector random variable and its value depends on ω: $\tilde{S}(T)=\tilde{S}(T,\omega)=\left(1, \tilde{S}_1(T,\omega), \ldots, \tilde{S}_N(T,\omega)\right)^T$.

A **portfolio** or a **trading strategy** h is an $(N+1)$-dimensional vector that describes the holdings of the investor, $h=\left(h_0, h_1, \ldots, h_N\right)$. Here h_n denotes the number of asset n held in the portfolio from the moment $t=0$ till terminal date T. The **value** of the portfolio at the moment t is given by

$$V^h(t)=h \cdot S(t)=\sum_{n=0}^{N} h_n S_n(t).$$

The value at the initial moment is a constant, whereas the value at the final moment is a random variable: $V^h(T,\omega) = h \cdot S(T,\omega) = \sum_{n=0}^{N} h_n S_n(T,\omega)$.

The **gain** from the portfolio h equals

$$G^h = V^h(T) - V^h(0) = h \cdot \Delta S = \sum_{n=0}^{N} h_n \Delta S_n,$$

where $\Delta S_n = S_n(T) - S_n(0)$ and $\Delta S = (\Delta S_0, \ldots, \Delta S_N)^T$. We define also the **discounted value** of the portfolio – i.e. the value of the portfolio measured in the units of the numéraire:

$$\tilde{V}^h(t) = \beta(t) V^h(t) = h \cdot \tilde{S}(t) = \sum_{n=0}^{N} h_n \tilde{S}_n(t).$$

The **discounted gain** is defined as

$$\tilde{G}^h = \tilde{V}^h(T) - \tilde{V}^h(0) = h \cdot \Delta\tilde{S} = \sum_{n=1}^{N} h_n \Delta\tilde{S}_n,$$

where $\Delta\tilde{S}_n = \tilde{S}_n(T) - \tilde{S}_n(0)$. Notice that summation starts with $n=1$ as the increment in the discounted value of the numéraire is 0.

Example 1

Assume that there are three assets. The numéraire is the bank account and the riskless interest rate equals 5%. Thus $S_0(0)=1$ and $S_0(T)=1.05$. The discount factor equals $\frac{1}{1.05}=0.95238$. The prices of risky assets at the initial moment are $S_1(0)=50$ and $S_2(0)=150$. We assume that there are three states of the world, so $\Omega = \{\omega_1, \omega_2, \omega_3\}$. We assume the following probabilities for the states of the world: $P(\omega_1)=\frac{1}{2}$ and $P(\omega_2)=P(\omega_3)=\frac{1}{4}$. The final prices of risky assets in different states of the world are given in Table 1.

The discounted prices of risky assets in all the possible states of the world are given in Table 2.

Table 1. Data for example 1–prices at the terminal date

	ϖ_1	ϖ_2	ϖ_3
$S_0(T,\omega)$	1.05	1.05	1.05
$S_1(T,\omega)$	47.25	47.25	63
$S_2(T,\omega)$	162.75	157.5	152.25

Table 2. Example 1–discounted prices at the terminal date

	ϖ_1	ϖ_2	ϖ_3
$\tilde{S}_0(T,\omega)$	1	1	1
$\tilde{S}_1(T,\omega)$	45	45	60
$\tilde{S}_2(T,\omega)$	155	150	145

The increments in the prices and discounted prices are given in Table 3.

Table 3. Example 1–increments in the prices and discounted prices

	ϖ_1	ϖ_2	ϖ_3
$\Delta S_0(\omega)$	0.05	0.05	0.05
$\Delta S_1(\omega)$	–2.75	–2.75	13
$\Delta S_2(\omega)$	12.75	7.5	2.25

	ϖ_1	ϖ_2	ϖ_3
$\Delta \tilde{S}_0(\omega)$	0	0	0
$\Delta \tilde{S}_1(\omega)$	–5	–5	10
$\Delta \tilde{S}_2(\omega)$	5	0	–5

A portfolio is a three-dimensional vector $h=(h_0,h_1,h_2)^T$. Its value at the initial moment equals $V^h(0)=h_0+50h_1+150h_2$. The value at the terminal moment depends on the state of the world. In Table 4 the values, discounted values, gains and discounted gains are calculated for every possible state of the world.

Table 4. Example 1–values and gains

	ϖ_1	ϖ_2	ϖ_3
$V^h(T,\omega)$	$1.05h_0+47.25h_1+162.75h_2$	$1.05h_0+47.25h_1+157.5h_2$	$1.05h_0+63h_1+152.25h_2$
$\tilde{V}^h(T,\omega)$	$h_0+45h_1+155h_2$	$h_0+45h_1+150h_2$	$h_0+60h_1+145h_2$
$G^h(\omega)$	$0.05h_0-2.75h_1+12.75h_2$	$0.05h_0-2.75h_1+7.5h_2$	$0.05h_0+13h_1+2.25h_2$
$\tilde{G}^h(\omega)$	$-5h_1+5h_2$	$-5h_1$	$10h_1-5h_2$

Example 2

We assume, as in Example 1, that interest rate is 5% and the prices of two risky assets at the initial date are 50 and 150. There are three equally possible states of the world and the prices at the terminal date are given in Table 5.

Table 5. Data for example 2–prices at the terminal date

	ϖ_1	ϖ_2	ϖ_3
$S_1(T,\omega)$	42	47.25	63
$S_2(T,\omega)$	162.75	157.5	152.25

Try to calculate discounted prices, increments and discounted increments as well as values and gains (and their discounted counterparts). The answers are given in Table 6 and Table 7.

Table 6. Discounted prices and increments in Example 2

	ϖ_1	ϖ_2	ϖ_3
$\tilde{S}_0(T,\omega)$	1	1	1
$\tilde{S}_1(T,\omega)$	40	45	60
$\tilde{S}_2(T,\omega)$	155	150	145
$\Delta S_0(\omega)$	0.05	0.05	0.05
$\Delta S_1(\omega)$	–8	–2.75	13
$\Delta S_2(\omega)$	12.75	7.5	2.25

	ϖ_1	ϖ_2	ϖ_3
$\Delta\tilde{S}_0(\omega)$	0	0	0
$\Delta\tilde{S}_1(\omega)$	–10	–5	10
$\Delta\tilde{S}_2(\omega)$	5	0	–5

Table 7. Values and gains in Example 2

	ϖ_1	ϖ_2	ϖ_3
$V^h(T,\omega)$	$1.05h_0+42h_1+162.75h_2$	$1.05h_0+47.25h_1+157.5h_2$	$1.05h_0+63h_1+152.25h_2$
$\tilde{V}^h(T,\omega)$	$h_0+40h_1+155h_2$	$h_0+45h_1+150h_2$	$h_0+60h_1+145h_2$
$G^h(\omega)$	$0.05h_0-8h_1+12.75h_2$	$0.05h_0-2.25h_1+7.5h_2$	$0.05h_0+13h_1+2.25h_2$
$\tilde{G}^h(\omega)$	$-10h_1+5h_2$	$-5h_1$	$10h_1-5h_2$

Example 3

Let us take all the data from Example 1 and assume that there is one additional state of the world ω_4. The sample space is now $\Omega=\{\omega_1,\ldots,\omega_4\}$. We assume that all outcomes are equally probable, thus $P(\omega_i)=\frac{1}{4}$. The prices of risky assets at the terminal date in the new state of the world are $S_1(T,\omega_4)=68.25$ and $S_2(T,\omega_4)=147$. The discounted prices in the additional state of the world are $\tilde{S}_1(T,\omega_4)=65$ and $\tilde{S}_2(T,\omega_4)=140$. The increments are $\Delta S_1(\omega_4)=18.25$, $\Delta S_2(\omega_4)=-3$, $\Delta\tilde{S}_1(\omega_4)=15$ and $\Delta\tilde{S}_2(\omega_4)=-10$. The value and discounted value are $V^h(T,\omega_4)=1.05h_0+68.25h_1+147h_2$ and $\tilde{V}^h(T,\omega_4)=h_0+65h_1+140h_2$.

The gain and discounted gain equal $G^h(\omega_4)=0.05h_0+18.25h_1-3h_2$ and $\tilde{G}^h(\omega_4)=15h_1-10h_2$.

3.1.2. Arbitrage opportunity and the First Fundamental Theorem of Asset Pricing

We assume that the market is "frictionless". There are no transaction costs or taxes, an investor can build any portfolio he wishes – there are no restrictions on the size of position, unlimited short sales and borrowing are allowed. Additionally, the securities are perfectly divisible, which means that the investor's positions h_i can take any real values.

To be economically reasonable, the model should fulfill some additional assumptions. In particular, the model is unreasonable if it assumes that the investor is able to make profits without any exposure to risk. Such a possibility would be "a free lunch" and it is assumed that it is impossible in the real market. Formally, we define this as arbitrage opportunity.

A portfolio h is an **arbitrage opportunity** (or **arbitrage strategy**) if its initial value is zero, $V^h(0)=0$, and its value at the terminal date satisfies

$$P\left(V^h(T)\geq 0\right)=1 \quad \text{and} \quad P\left(V^h(T)>0\right)>0.$$

The arbitrage opportunity is thus a portfolio with an initial value, which almost surely (i.e. with probability 1) produces a non-negative final value and, with positive probability, its final value can be positive. In this definition the nominal (not discounted) values were used. However, one can also use a discounted value or discounted gain to define the arbitrage opportunity.

In the model there exists an arbitrage opportunity if and only if there exists a portfolio h such that $\tilde{V}^h(0)=0$, and at the terminal date its discounted value fulfills:

$$P\left(\tilde{V}^h(T)\geq 0\right)=1 \quad \text{and} \quad P\left(\tilde{V}^h(T)>0\right)>0$$

or there is a portfolio h such that

$$P\left(\tilde{G}^h\geq 0\right)=1 \quad \text{and} \quad P\left(\tilde{G}^h>0\right)>0.$$

The last criterion, which uses discounted gain, is the easiest one to check. Note that in this criterion there is no assumption that initial value of the arbitrage strategy should equal zero. If there is no arbitrage opportunity in the model, we say that the model is **arbitrage-free** or **viable**.

Example 1 – cont.

We will consider if there exists an arbitrage opportunity in the model presented in Example 1. To this end we will make use of the last criterion based on discounted gain. Discounted gains in all states of the world are calculated in Table 3 (the last row). As all states of the world have positive probability, for the arbitrage strategy we should have $\tilde{G}^h(\omega) \geq 0$ for every ω. An arbitrage opportunity exists if and only if the following system of inequalities

$$-5h_1 + 5h_2 \geq 0,$$

$$-5h_1 \geq 0,$$

$$10h_1 - 5h_2 \geq 0$$

has a solution with at least one inequality being strict. As it is easy to check, the only solution to this system is $h_1 = h_2 = 0$. Thus in this model there is no arbitrage opportunity.

Example 2 – cont.

One can easily check that for any portfolio with $h_1 = -1$ and $h_2 = -2$ we have $\tilde{G}^h(\omega_1) = \tilde{G}^h(\omega_3) = 0$ and $\tilde{G}^h(\omega_2) = 5 > 0$. Thus the arbitrage opportunity exists in this model. If we take $h_0 = 350$, we obtain the portfolio $h = (350, -1, -2)$ such that $V^h(0) = 0$, $V^h(T, \omega_1) = V^h(T, \omega_3) = 0$ and $V^h(T, \omega_2) = 5.25$. This portfolio is thus an arbitrage strategy.

Let us consider the same model of market as before, but assume that instead of real probabilities $P(\omega)$ of different states of the world, we have some artificial probabilities $Q(\omega)$. We assume that the two probabilistic measures, P and Q, are equivalent, which means that $Q(\omega) > 0$ for all the states of the world with $P(\omega) > 0$. In our setup this is equivalent to the assumption that $Q(\omega) > 0$ for all $\omega \in \Omega = \{\omega_1, \ldots, \omega_M\}$.

A probabilistic measure Q is called a **martingale measure** (or **risk neutral measure**) if the initial prices of all instruments are equal to the expected values (calculated with respect to Q) of their discounted terminal prices:

$$S_n(0) = E^Q\left[\tilde{S}_n(T)\right] = E^Q\left[\beta(T)S_n(T)\right] = \sum_{i=1}^{M} Q(\omega_i)\beta(T)S_n(T, \omega_i). \quad (1)$$

In the formula above $E^Q[\cdot]$ is an expected value calculated for the probability measure Q. It should be noted that the martingale measure is always connected with the numéraire that is used. We can always change the numéraire and, if there exists a martingale measure for standard numéraire S_0, there is also a probability measure in which initial prices of all securities (measured in the units of numéraire) are expected values of their prices. For example, suppose that asset 1 can serve as a numéraire (i.e. its final prices is always positive, $S_1(T) > 0$). Let us define probability measure Q^1 as follows:

$$Q^1(\omega) = \frac{S_1(T,\omega)}{S_1(0)S_0(T,\omega)} Q(\omega). \tag{2}$$

One can easily check that probabilities Q^1 are positive and that they sum up to 1:

$$\sum_{i=1}^{M} Q^1(\omega_i) = \sum_{i=1}^{M} \frac{S_1(T,\omega_i)}{S_1(0)S_0(T,\omega_i)} Q(\omega_i) = \frac{1}{S_1(0)} \sum_{i=1}^{M} \tilde{S}_1(T,\omega_i) Q(\omega_i) = \frac{E^Q\left[\tilde{S}_1(T,\omega)\right]}{S_1(0)} =$$
$$= \frac{S_1(0)}{S_1(0)} = 1.$$

If we denote by $E^1[\cdot]$ the expected value calculated for the probability measure Q^1, we can make the following derivations:

$$S_1(0)E^1\left[\frac{S_n(T)}{S_1(T)}\right] = S_1(0)\sum_{i=1}^{M} \frac{S_n(T,\omega_i)}{S_1(T,\omega_i)} Q^1(\omega_i) =$$
$$= S_1(0)\sum_{i=1}^{M} \frac{S_n(T,\omega_i)}{S_1(T,\omega_i)} \frac{S_1(T,\omega_i)}{S_1(0)S_0(T,\omega_i)} Q(\omega_i) =$$

$$= \sum_{i=1}^{M} \frac{S_n(T,\omega_i)}{S_0(T,\omega_i)} Q(\omega_i) = E^Q\left[\tilde{S}_n(T)\right] = S_n(0).$$

Thus we have:

$$\frac{S_n(0)}{S_1(0)} = E^1\left[\frac{S_n(T)}{S_1(T)}\right],$$

which is an equivalent of equation (1) for the asset 1 as a numéraire. The probabilities Q^1 are called martingale measure for the numéraire S_1. Similarly, one can define martingale measures Q^2, Q^3, … and so on–for any security that can serve as a numéraire. The martingale measure Q, connected with the standard numéraire S_0 will be called simply "risk-neutral measure" or "martingale measure" (without any addition).

There exists a deep relationship between the existence of arbitrage opportunity in the model and martingale measures. It is known as "First Fundamental Theorem", which states that existence of an arbitrage opportunity and existence of martingale measure are mutually exclusive.

First Fundamental Theorem. The model of financial market is arbitrage-free if and only if there exists a martingale measure, i.e. the probability measure Q for which the following equation holds for all securities:

$$S_n(0)=E^Q\left[\frac{S_n(T)}{S_0(T)}\right]=E^Q\left[\beta(T)S_n(T)\right]. \tag{3}$$

This rule can be also reformulated as follows. The model is arbitrage-free if for any security S_m that can serve as a numéraire there exists a probability measure Q^m such that

$$S_n(0)=S_m(0)E^m\left[\frac{S_n(T)}{S_m(T)}\right]. \tag{4}$$

The economic interpretation of equations (3) and (4) is that initial prices of securities are obtained as the expected value (under appropriate probability measure Q or Q^m) of the final prices of assets, discounted with the chosen numéraire. The equation (3) is also referred to as the "risk-neutral" pricing formula. The proof of the theorem is rather simple and is based on the mathematical result known as "separating hyperplane theorem". The proof can be found, e.g. in (Bingham & Kiesel, 2004; Elliott & Kopp, 1999; Pliska, 1997).

The other question is how to find martingale probabilities, if such a measure exists in the model. One can try to calculate them directly from the definition – using equations (1) or (3). We can also transform these equations. Using the definition of increments of discounted prices, $\Delta\tilde{S}_n=\tilde{S}_n(T)-\tilde{S}_n(0)$, we obtain the following condition:

$$E^Q\left[\Delta\tilde{S}_n\right]=0 \quad \text{for all } n=1,\dots,N. \tag{5}$$

it. Moreover, as we pointed out in the previous subchapter, public companies may offer securities not only to raise additional capital but for many different reasons. They still can use all three methods: rights offering (when they offer new shares to existing shareholders, e.g. to avoid a hostile takeover); private placements (especially when they issue bonds to a small group of financial institutions) and subsequent public offerings (when they offer newly issued shares on the open market to raise capital).

Public offerings are more complex and more time-consuming than private placements. Newly issued stocks or bonds that are **publicly offered have to be registered** by a given regulatory authority, such as Securities and Exchange Commission (SEC) in the U.S. or Polish Financial Supervision Authority (KNF) in Poland. Not all the companies meet the requirements set by regulators to register their securities, so this type of offering may be not available to many companies.

In public companies the **decision about new security issue** and its form is made **by managers** (executives) but in some legal systems it has to be approved by existing shareholders. Generally, in the U.S. managers (or being precise–a board of directors) have far-reaching flexibility in issuing new securities. Moreover, the power of deciding to whom the newly issued shares are offered and what type of offering is used is practically in the hands of the board of directors, mostly because of **limitations of shareholders' preemptive rights**. In Europe, on the contrary, **shareholders** must **approve managers' propositions**. Their preemptive rights are generally given as a general principle (the so-called default rule). Moreover, in some countries any increase in stock capital has to be approved by shareholders' meeting with **supermajority**, so managers' discretion is thus curbed to a significant extent.

4.5. Types of offerings

4.5.1. Private placements

As we mentioned before, capital market regulatory and supervisory authorities register all publicly offered securities. The registration procedure may be quite complex, costly and time-consuming. This does not refer to private placements, which is the biggest advantage of this method of security offering. Offerings are treated as private placements when they target a **specified and limited group of investors** Currently (2018) the maximum acceptable number of targeted investors equals **35** in the U.S. and **149** in Poland. An offer to an undefined group of investors or an offer to a defined group of investors exceeding a certain number (e.g. to 150 or more investors in Poland) is treated as a public offering.

Private placements are generally much **easier to arrange** and require **much less information** to be revealed by the issuing company. They are also less time-consuming than public offerings, which means that additional **capital can be raised relatively quickly**. Moreover, companies' managers (in private companies simply their owners) may decide to whom new securities are offered so they can "set" the ownership structure (to some extent it is also possible in public offerings but with limitations).

The main disadvantage of private placements is a **limited secondary market**. Any investor deciding to buy bonds or stocks in a private placement should be aware that it may be quite difficult to exit, because it is not so easy to find a buyer outside the open market. It is a problem for investors but also for issuing companies – to attract investors they have to compensate them for this lack of liquidity by offering **extra premium** to the rate of return. That's why it is not obvious whether private placements are "effectively" cheaper than public offerings. On the one hand, direct costs of organizing an offer (fees for legal advisors, auditors and underwriters, commissions paid to authorities and stock exchanges, etc.) are lower but, on the other hand, cost of capital (e.g. an offered coupon rate in bond offering) is higher.

Private placements are **used mostly for debt instruments**, especially in the U.S., where bonds are offered to institutional investors, like pension funds, mutual funds or insurance companies. Notice that bonds have strictly defined maturity, so these investors will be eventually paid off when the bonds mature without a need to resell them (of course, they will have to find a buyer, if they want to exit before maturity date). It does not refer to stocks which have undefined maturity, so to exit, one needs to find a willing buyer.

Since the '90s this method has been also used extensively by mature European companies and companies from other parts of the world to issue both stocks and bonds in the U.S. due to some relaxed restrictions in subsequent trading of such securities. In 1990 SEC adopted **Rule 144A** under which securities can be offered in the form of a private placement to U.S. big financial institutions (with at least $100m in assets under management), called **qualified institutional buyers** (**QIBs**), who can trade unregistered securities among themselves before they are eventually traded on the open market. Relaxing the trading restrictions significantly broadened the US private equity market (and bond market, especially for non-U.S. companies).

Polish listed companies use private placements mainly to raise additional capital in a special form called target capital (unknown in common law systems). Shareholders can delegate to managers the power to increase the initial share capital by the amount that cannot exceed 75% of the outstanding shares (similar rules exist in other European countries, including Germany, Italy and many more but they can differ slightly). Managers can exercise this right and increase the share capital by issuing and offering new shares to old or new shareholders. In practice, they typically start with offering subscription warrants to a specified group of investors. These warrants can be eventually converted into company's shares. The procedure allows to raise capital relatively quickly with the transaction costs limited to the minimum, which is its main advantage, as pointed out by managers of public companies listed on the Warsaw Stock Exchange. It gives managers valuable financial flexibility, but at the cost of limited shareholders' rights. Minority investors often claim that the procedure leads to the dilution effect because it allows to exclude their preemptive rights (this is one of the exceptions from the general principle when current shareholders cannot exercise their preemptive rights to new shares).

4.5.2. Initial public offerings

4.5.2.1. Basics of IPOs and reasons for going public

Initial public offering (IPO) is a company's first security offering to the public (unknown investors). It may refer to both stocks and bonds, however, typically one means stock offering when talking about IPO so we will also do this throughout this chapter.

After the IPO a private (closed) company becomes a **public company** (also called a **listed company**), because from that time on its shares (bonds) are traded on an open market and thus are easily available to the public. **Going public** (called also simply **debut** or **flotation**) is typically a milestone in company's life cycle.

First of all, it causes a big change in company's **ownership structure**. After the IPO the number of company's owners may increase from several founders to thousands or even millions of stockholders. It obviously makes the ownership structure of public companies much more dispersed, meaning that an average stockholder holds a relatively small stake in company's capital. Old shareholders lose some of their voting power but typically retain control over the company even when they sell part of their stake during IPO. Moreover, it is not only the number of shareholders that grows after IPO, but also the frequency the stakes trade between subsequent stockholders. The ownership structure of public companies is changing all the time as millions of its shares can be traded daily on the open market.

Secondly, a public company needs to change totally its attitude towards disclosures. Closely held companies typically keep all the information about their activities secret, unless some information is required by special rules (it may refer to financial statements informing about companies' financial performance and financial position). Public companies, on the other hand, have to meet

mandatory disclosure and reporting standards, meaning that they have to reveal all important information (information that could affect its market price) to the public. It is required by regulatory authorities to give all the outside investors the same opportunity to use the information in their investment decisions. Actually, the reporting requirements for public companies imposed by regulatory authorities are the main reason why the owners of many big, mature companies still prefer their firms to operate as closely-held companies.

We started this chapter with indicating the main aims of security offerings highlighting the company's need for additional capital. In the section devoted to IPOs we should add some special aims of this special form of offering. Why do companies go public? Of course, most of them do this because listed companies have better access to capital. It is why relatively young firms with substantial capital need to go public, especially in the U.S. where the average age of such companies is much lower than in Europe.

Nevertheless, it is definitely not the only reason why companies organize IPOs. First of all, main shareholders often use an IPO as the opportunity to **cash out**, an important reason which we also mentioned earlier. It refers especially to venture capital and private equity firms (**VC/PE** firms) that invest in young private companies, sometimes at very early stage (e.g. startup) and exit often via IPOs. Other group of investors that often cash out during IPOs are companies' founders. However, founders typically sell a relatively small part of their stakes, contrary to VC/PE firms that basically get rid of their whole stake in companies going public.

In practice

Dino Polska SA is a Polish retailer that went public in 2017. It was one of the biggest IPOs in the last several years. A secondary tranche (the only one) consisted of shares held (indirectly) by one of the biggest private equity firm that operates in Poland – Enterprise Investors. They were sold for PLN 33.50, which gives an offer the size of PLN 1.6bn. Enterprise Investors had invested about PLN 200m seven years before the company went public. It gives a holding period pre-tax rate of return of app. 700% (no dividends were paid during that period).

IPOs are sometimes used also to establish market price of stock and to let the market asses company's performance. This reason is highlighted when companies engage in **equity carve-out** transactions. Equity carve-out is a form of a divestiture when a company sells to the public its stake in a subsidiary, typically a business segment previously separated from the parent company that operates in different industry.

Last, but not least, managers often claim that one of the reasons for organizing an initial public offering and going public is to gain publicity and enhance company's reputation. It is not easy to asses if there is any "marketing" gain from going public, but it seems reasonable that it can enhance companies' credibility, mainly due to the transparency forced by reporting standards.

4.5.2.2. IPO procedures and requirements

Decision to go public. The very first step in IPO is always the decision to go public, which is typically proposed by managers and approved by shareholders (notice that in many private, closely held companies it may be the same individuals). The formal approval may take the form of a General Shareholders' Meeting's resolution. It is worth mentioning that those firms that operate in another form than a corporation (e.g. partnerships) have to **incorporate** before they go public. Not all the companies are eligible to go public. They have to meet the criteria set by stock exchanges, called **listing requirements** covering generally three areas: **financial history, size** and **liquidity**. Many stock exchanges require companies to operate for at least 3-5 years before going public. Most exchanges also require minimum earnings level or **minimum market capitalization** (total number of shares outstanding multiplied by the share market price). Companies have to submit financial statements documenting their financial performance to regulatory authorities and to stock exchanges for approval. Stock exchanges care about future trading so they require minimum liquidity guaranteed by the company. It is typically measured with the so-called **free float** which is the proportion of outstanding stocks available for trading after all blockholders' stakes are excluded.

In practice

In many countries younger and smaller companies that cannot be listed in the ***main markets*** *may still choose* ***sub-markets*** *with less strict requirements. The best examples are* ***AIM*** *(Alternative Investment Market) run by the London Stock Exchange with 960 companies listed at the end of 2017 and total capitalization exceeding GBP 100bn or* ***New Connect*** *run by the Warsaw Stock Exchange with 400 companies and total capitalization of approximately PLN 10bn. It is worth mentioning that some companies listed in sub-markets implement relatively poor corporate governance structures and even stop meeting the limited listing requirements (e.g. stop publishing financial reports) and are eventually excluded from listings. Sub-markets are generally full of the so-called* ***penny stocks****. Deutsche Boerse was forced to close its sub-market,* ***Neuer Markt****, in 2002 to improve market transparency and regain investor confidence.*

Choosing underwriter(s) and other advisors. After the decision is made, the company should choose advisors and auditors. They are responsible for **business** and **legal due diligence** of the company. The most important role is played