The difference of two relations is formed by the pairs, which are in the first relation, but not in the other. Consider, for example, the difference $P \setminus Q$, for example the pairs starting with 2. As we can see, among the members of P there are only the pairs (2, 1) and (2, 2) and in the case of Q – the pairs (2, 2), (2, 3) and (2, 4). Of these, only the pair (2, 1) belongs to P and not to Q. If we refer to the matrix representation, in row "2" only the 1 in column "1" is present in the matrix of P, and not in the matrix of Q. Similarly, in row "2" only 1 in the columns "3" and "4" are present in the matrix of Q and not in P. Analyzing in a similar way the other rows, we obtain the matrices of differences:

$$\begin{array}{c} P \setminus Q \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad \begin{array}{c} Q \setminus P \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

To create the matrix of inverse relation, we rewrite the rows of the matrix as columns (or columns as rows)[4]. For example, the first row of the matrix of P consists of elements 1, 0, 0, and 0, so in the same order they form a first column of the matrix of P^{-1}. We apply the same operation to the other rows of the matrices of relations P and Q, which results with the matrices of the inverse relations. Please note that the relations Q and Q^{-1} are equal.

$$\begin{array}{c} P^{-1} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \qquad \begin{array}{c} Q^{-1} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Example 4

Let $P \subset B \times C : xPy \Leftrightarrow x + y \leq 5$ and $Q \subset A \times B : xQy \Leftrightarrow x \cdot y \geq 5$, where $A = \{1, 2, 3\}$, $B = \{2, 3, 4, 5\}$ and $C = \{0, 2, 4\}$. Write them in the form of matrices and find the composition $P \circ Q$.

[4] This operation is called **transposition**. You will find more about transposition in the next chapter.

Solution
The matrices look as follows:

$$\begin{array}{c|c} P & \begin{matrix} 0 & 2 & 4 \end{matrix} \\ \begin{matrix} 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array} \qquad \begin{array}{c|c} Q & \begin{matrix} 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

By the definition of composition, it is a subset of $A \times C$ in this case. It consists of all the pairs (x, y) such that there is at least one $z \in B$ such that xQz and zPy. Verifying all the members of $A \times C$ may however take too much time, so we will use another method. We are looking for the composition $P \circ Q$, so for each element x being member of $D(Q)$ (i.e., of the domain of Q, in this case being equal to A) we check to which members of $D^{-1}(Q)$ (counterdomain of Q, i.e., B) it is Q-related. For each such element we check in turn, to which elements of the counterdomain of P (i.e., C), it is P-related. And those are the elements to which x is $(P \circ Q)$-related. The phases of determining the matrix of $P \circ Q$ are presented below.

$$\begin{array}{c|c} P \circ Q & \begin{matrix} 0 & 2 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 \\ & & \\ & & \end{bmatrix} \end{array} \qquad \begin{array}{c|c} P \circ Q & \begin{matrix} 0 & 2 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ & & \end{bmatrix} \end{array} \qquad \begin{array}{c|c} P \circ Q & \begin{matrix} 0 & 2 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

Let us consider all the members of $D(Q)$. Let us start with 1. It is Q-related only to 5 (in the matrix of Q in the row "1" the only 1 is in the column "5"). Thus we find the row "5" in the matrix of P and we copy each 1 that we find. There is only one such 1, in the column "0". We copy it to the matrix of the composition, filling the rest of the row with 0 (left matrix above). We step to the next member of $D(Q)$, i.e., 2. In the row "2" of the matrix of Q we find 1 in the columns "3", "4" and "5", so we check the rows "3", "4" and "5" in the matrix of P. There are two 1 in the row "3": in the columns "0" and "2". In the row "4" there is only one 1 in the column "0". In the row "5" there is also only one 1 in the column "0". Finally, we have 1 in the columns "0" and "2" (it does not matter that in the column "0", 1 appears three times). For that reason, we insert 1 in the matrix of the composition only to the columns "0" and "2", and the remaining elements of the row "2" we fill with 0 (actually, we put only one 0 in the column "4"). The effect of our actions can be seen in the central matrix above. The last member of $D(Q)$ is 3. It is Q-related to 2, 3, 4 and 5. We check respective rows in the matrix of P and we find out that taking them all into account we can insert 1 only to the columns "0" and "2" (both copies of 1 in the rows "2" and "3", moreover one 1 in the column "0" in the rows "4" and "5" of the matrix of P). Thus in the row "3" of the matrix of the composition there will be 1 in the columns "0" and "2". We see the final result in the last matrix above.

Example 5

Let $P \subset A^2 : xPy \Leftrightarrow x \geq y$ and $Q \subset A^2 : xQy \Leftrightarrow x + y \geq 4$, where $A = \{1, 2, 3, 4\}$. Find the compositions $P \circ Q$, $Q \circ P$, $P \circ P$ and $Q \circ Q$.

Solution

The matrices of P and Q are as follows:

$$\begin{array}{c|c} P & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array} \qquad \begin{array}{c|c} Q & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

In each of these cases we find the composition the same way as in the previous example. Note only that the two relations are defined on a product of A with itself, so also each of the compositions be a subset of A^2.

In the case of a $P \circ Q$ we check first which elements are Q-related to which, and then which are P-related to which. For example, let us consider 1. It is Q-related to 3 and 4. We move to the matrix of P, and we find that in the rows "3" and "4" 1 can be found in each column. So finally 1 is ($P \circ Q$)-related to all the elements of A. Similarly we proceed with the composition $Q \circ P$, but here we first check the relation P, then Q. For example, 1 is P-related only to 1, which is Q-related to 3 and 4. This means that 1 is $P \circ Q$-related to 3 and 4. The final effect has been presented below:

$$\begin{array}{c|c} P \circ Q & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array} \qquad \begin{array}{c|c} Q \circ P & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

The compositions $P \circ P$ (i.e., P^2) and $Q \circ Q$ (i.e., Q^2) we find similarly, analyzing twice the matrix of P (or Q, respectively):

$$\begin{array}{c|c} P^2 & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array} \qquad \begin{array}{c|c} Q^2 & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

EXERCISES

1. Present the following relations in the form of a list of pairs, a matrix and a graph (use the simplified version of graph where it is possible).
   a) $P \subset A \times B : xPy \Leftrightarrow x + y \leq 3$, $A = \{-2, -1, 0, 1, 2\}$, $B = \{1, 2, 3\}$;
   b) $P \subset A^2 : xPy \Leftrightarrow \exists\, k \in Z : x - y = 2k$, $A = \{-2, -1, 0, 1, 2\}$;
   c) $P \subset A \times B :\quad xPy \Leftrightarrow \exists\, k \in Z : x - y = 3k$, $\quad A = \{x \in N : x < 2\}$, $B = \{x \in N : x < 4\}$;
   d) $P \subset A^2 : xPy \Leftrightarrow x^2 < y^2$, $A = \{x \in Z : -1 \leq x \leq 3\}$.
2. Present the matrices of the following relations and then find their complements, unions, intersections, differences and inverse relations (where it is possible):
   a) $P \subset A \times B : xPy \Leftrightarrow x + y \leq 2$, $Q \subset A \times B : xQy \Leftrightarrow x - y \geq -1$, $A = \{-2, -1, 0, 1, 2\}$, $B = \{1, 2, 3\}$;
   b) $P \subset A^2 : xPy \Leftrightarrow \exists\, k \in Z : x - y = 2k$, $Q \subset A^2 : xQy \Leftrightarrow xy > 0$, $A = \{-1, 0, 1, 2\}$;
   c) $P \subset A \times B : xPy \Leftrightarrow xy \leq 2$, $Q \subset B \times C : xQy \Leftrightarrow xy \geq 1$, $A = \{-1, 0, 1, 2\}$, $B = \{1, 2, 3\}$, $C = \{x \in N : x < 2\}$.
3. Find all the existing compositions among the following: $P \circ Q$, $Q \circ P$, P^2 and Q^2:
   a) $P \subset A \times B : xPy \Leftrightarrow x + y \leq 2$, $\qquad Q \subset A \times B : xQy \Leftrightarrow x - y \geq -1$, $A = \{-2, -1, 0, 1, 2\}$, $B = \{1, 2, 3\}$;
   b) $P \subset A^2 : xPy \Leftrightarrow \exists\, k \in Z : x - y = 2k$, $Q \subset A^2 : xQy \Leftrightarrow xy > 0$, $A = \{-1, 0, 1, 2\}$;
   c) $P \subset A \times B : xPy \Leftrightarrow xy \leq 2$, $Q \subset B \times C : xQy \Leftrightarrow xy \geq 1$, $A = \{-1, 0, 1, 2\}$, $B = \{1, 2, 3\}$, $C = \{x \in N : x < 2\}$.

SOLUTIONS

1.
   a) $P = \{(-2, 1), (-2, 2), (-2, 3), (-1, 1), (-1, 2), (-1, 3), (0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (2, 1)\}$;

the total profit from the production of P_1 will be $10x_1$ (profit from production of one piece equal to 10 PLN times the number of units, denoted by x_1). Similarly, the profit of the manufacture of the product P_2 is $9x_2$. The total profit is therefore:

$f(x_1, x_2) = 10x_1 + 9x_2$.

Let us now turn to the resources. Consumption of S_1 per unit of product P_1 is 3 kg. Hence, the total consumption of this resource to the production of P_1 is equal to $3x_1$ (consumption per piece equal to 3 kg times the number of pieces, denoted by x_1). Similarly, we calculate the consumption of S_1 to produce P_2: $6x_2$. Finally the total consumption of S_1 is $3x_1 + 6x_2$. Consumption cannot exceed the limit which is equal to 1500 kg. This can be written in the form of inequality: $3x_1 + 6x_2 \leq 1500$. Similarly we can write the constraint corresponding with the second resource: $4x_1 + 3x_2 \leq 1200$. Remember also that the variables must be nonnegative (you cannot produce a negative number of products). Considering the above, the corresponding the linear programming problem is as follows:

(0) $f(x_1, x_2) = 10x_1 + 9x_2 \to \max$,
(1) $3x_1 + 6x_2 \leq 1500$,
(2) $4x_1 + 3x_2 \leq 1200$,
(3) $x_1, x_2 \geq 0$.

After introducing the slack variables it takes the form:

$$\begin{cases} 3x_1 + 6x_2 + x_3 = 1500, \\ 4x_1 + 3x_2 + x_4 = 1200, \end{cases}$$

and the augmented matrix is:

$$\left[\begin{array}{cc|cc|c} 3 & 6 & 1 & 0 & 1500 \\ 4 & 3 & 0 & 1 & 1200 \end{array}\right].$$

The rank of the matrix is 2, so all the bases will consists of two elements. All the bases, the corresponding basic solutions and objective values (OBJ) have been presented in Table 5.5.

Table 5.5. Basic solutions and the objective (OBJ)

Base	Solution	OBJ
$B^{(1)} = \{x_1, x_2\}$	$x^{(1)} = (180, 160, 0, 0)$	3240
$B^{(2)} = \{x_1, x_3\}$	$x^{(2)} = (300, 0, 600, 0)$	3000
$B^{(3)} = \{x_1, x_4\}$	$x^{(3)} = (500, 0, 0, -800)$	×
$B^{(4)} = \{x_2, x_3\}$	$x^{(4)} = (0, 400, -900, 0)$	×
$B^{(5)} = \{x_2, x_4\}$	$x^{(5)} = (0, 250, 0, 450)$	2250
$B^{(6)} = \{x_3, x_4\}$	$x^{(6)} = (0, 0, 1500, 1200)$	0

As we can see, the maximum value of the objective corresponds to $x^{(1)}$: $f_{\max}(180, 160) = 3240$. It follows that in order to achieve the maximum weekly profit, equal to 3240 PLN, the company should produce 180 pieces of P_1 and 160 pieces of P_2 weekly.

Example 3

The cat can be fed with dry food or canned food. The water and protein content in the two types of food (grams per 100 g), their prices (PLN per 100 g) and the required daily intake of both components (in grams) are given in Table 5.6. Determine the daily cat diet with minimal cost.

Table 5.6. Information about food and contents

Content \ Food	Canned	Dry	Required intake
Protein	10	40	30
Water	80	0	80
Price per 100 g	3	2	

Solution

This time the decision variables will be: x_1 – the amount of 100 g – servings of the canned food in the daily cat diet, x_2 – the amount of 100 g – servings of the dry food in the daily cat diet. The daily cost will be: $3x_1 + 2x_2$, the protein intake $10x_1 + 40x_2$, and the water intake – $80x_1$. The intake of both ingredients must be at least as high as the required amount, so both inequalities will take the form "≥". The cost of the food should be as small as possible. This means that the corresponding linear programming problem is:

(0) $f(x_1, x_2) = 3x_1 + 2x_2 \to \min$,

(1) $10x_1 + 40x_2 \geq 30$,

(2) $80x_1 \geq 80$,

(3) $x_1, x_2 \geq 0$.

After introducing the slack variables it takes the form:

$$\begin{cases} 10x_1 + 40x_2 - x_3 \qquad = 30, \\ 80x_1 \qquad\qquad\quad - x_4 = 80, \end{cases}$$

and the augmented matrix is (note that its rank equals to 2):

$$\left[\begin{array}{cc|cc|c} 10 & 40 & -1 & 0 & 30 \\ 80 & 0 & 0 & -1 & 80 \end{array}\right].$$

The basic solutions and corresponding values of the objective are presented in Table 5.7.

Table 5.7. Basic solutions and the objective (OBJ)

Base	Solution	OBJ
$B^{(1)} = \{x_1, x_2\}$	$x^{(1)} = (1, 1/2, 0, 0)$	4
$B^{(2)} = \{x_1, x_3\}$	$x^{(2)} = (1, 0, -20, 0)$	×
$B^{(3)} = \{x_1, x_4\}$	$x^{(3)} = (3, 0, 0, 160)$	9
$B^{(4)} = \{x_2, x_3\}$	contradiction	×
$B^{(5)} = \{x_2, x_4\}$	$x^{(5)} = (0, 3/4, 0, -80)$	×
$B^{(6)} = \{x_3, x_4\}$	$x^{(6)} = (0, 0, -30, -80)$	×

The smallest value of the objective corresponds to $x^{(1)}$: $f_{\min}(1, 1/2) = 4$. It means that in order to achieve the minimum cost of feeding the cat, equal to 4 PLN, it is necessary to give him every day 1 serving (100 g) of canned food and 1/2 serving (50 g) of dry food.

EXERCISES

1. Solve the linear programming problems:

   a) (0) $f(x_1, x_2) = 5x_1 + 6x_2 \to \max$,

   (1) $4x_1 + x_2 \le 20$,

   (2) $2x_1 + 8x_2 \le 40$,

   (3) $x_1, x_2 \ge 0$;

   b) (0) $f(x_1, x_2) = 5x_1 + 6x_2 \to \min$,

   (1) $4x_1 + x_2 \ge 20$,

   (2) $2x_1 + 8x_2 \ge 40$,

   (3) $x_1, x_2 \ge 0$;

c) (0) $f(x_1, x_2) = 3x_1 + 10x_2 \to \max$,
(1) $2x_1 + \ \ x_2 \le 24$,
(2) $3x_1 + 6x_2 \le 72$,
(3) $x_1, x_2 \ge 0$;

d) (0) $f(x_1, x_2) = 3x_1 + 10x_2 \to \min$,
(1) $2x_1 + \ \ x_2 \ge 24$,
(2) $3x_1 + 6x_2 \ge 72$,
(3) $x_1, x_2 \ge 0$.

2. The company produces two products using two resources. Data on unit profit (in thousand PLN), consumption of resources (in kg per piece), and monthly resource limits (kg) are presented in the table. Determine the production plan maximizing the profit of the company.

Product / Resource	P_1	P_2	Limit
S_1	5	10	150
S_2	20	10	300
Profit	4	3	

3. The company produces two products using two resources. Data on unit profit (in thousand PLN), consumption of resources (in kg per piece), and monthly resource limits (kg) are presented in the table. Determine the production plan maximizing the profit of the company.

Product / Resource	P_1	P_2	Limit
S_1	5	15	240
S_2	30	10	480
Profit	3	4	

4. The diet consists of two food products that comprise, among others, two nutrients. Data on prices of products (in PLN per 100 g), the content of the ingredients in the products (g per 100 g) and the weekly norm of nutrient intake (g) are presented in the table. Find a diet with minimal cost.

Product / Nutrient	P_1	P_2	Required amount
S_1	5	10	150
S_2	20	10	300
Price	4	3	

5. The diet consists of two food products that comprise, among others, two nutrients. Data on prices of products (in PLN per 100 g), the content of the ingredients in the products (g per 100 g) and the weekly norm of nutrient intake (g) are presented in the table. Find a diet with minimal cost.

Product / Nutrient	P_1	P_2	Required amount
S_1	5	15	240
S_2	30	10	480
Price	3	4	

SOLUTIONS

1. a) $f_{\max}(4, 4) = 44$; b) $f_{\min}(4, 4) = 44$; c) $f_{\max}(0, 12) = 120$; d) $f_{\min}(24, 0) = 72$.
2. x_1 – the volume of production of the product P_1 expressed in pieces, x_2 – the volume of production of the product P_2 expressed in pieces; the model:

   (0) $f(x_1, x_2) = 4x_1 + 3x_2 \to \max,$

   (1) $5x_1 + 10x_2 \le 150,$

   (2) $20x_1 + 10x_2 \le 300,$

   (3) $x_1, x_2 \ge 0.$

   Optimal volume of production: 10 pc. of P_1 and 10 pc. of P_2, maximum profit: 70 000 PLN.

3. x_1 – the volume of production of the product P_1 expressed in pieces, x_2 – the volume of production of the product P_2 expressed in pieces; the model:

causes 0.5% increase of demand. When the price is 10 PLN and the advertising expenses are 25 thousand PLN, the demand equals to 100 thousand pieces. Find the demand function. What should be the price if the advertising expenses are 100 thousand PLN and we want the demand to be 400 thousand pieces?

Solution

Taking into account the constraints, in a general case we should solve the relevant initial value problem. However, since the function has constant elasticities (–1 with respect to the price and 0.5 with respect to the advertising expenses), we can be sure that it is a Cobb–Douglas function. Its formula is: $D(P, R) = CP^{-1}R^{0.5}$, where C is a constant. In order to find it we substitute the initial condition $D(10, 25) = 100$: $C\cdot 10^{-1}\cdot 25^{0.5} = 100$. Hence $C = 200$. Finally, the demand function is $D(P, R) = 200P^{-1}R^{0.5}$. In order to find the desired price, we substitute the demand and advertising expenses: $D(P, 100) = 400$, i.e. $200P^{-1}100^{0.5} = 400$, thus $P = 5$. The price should be equal to 5 PLN.

Example 5

The company has established the following relationship between the demand for margarine, measured in terms of sales S (tons per week), and the average income of consumers D (in PLN for 1 person per month) and the price of margarine C (in PLN per 1 kg): $S = 20\, D^{0.5}\, C^{-1.2}$. Also the relationship between the weekly production cost K (thousand PLN) and production volume Q (in tons per week) has been estimated: $K = 100 + 3Q$. We assume that the weekly production is adapted to the volume of sales. Current price of margarine $C_1 = 6$ PLN/kg, and the sales volume $S_1 = 50$ tons. Calculate the approximate change of demand for margarine, if its price falls by 10%. Determine how in this case will change weekly revenue, cost and profit.

Solution

Since power function describes the sales, its price elasticity is constant (equal to –1.2). It means that 10% decrease of price, will cause the demand decrease by about $1.2\% \cdot 10 = 12\%$. Thus the new price will be $6 - 10\% \cdot 6 = 5.4$ PLN, and the new demand $50 + 12\% \cdot 50 = 56$ tons. Before the change, the revenue was $P_1 = C_1{\cdot}S_1 = 6 \cdot 50 = 300$ thousand PLN, and the cost $K_1 = 100 + 3S_1 =$ $= 100 + 3 \cdot 50 = 250$ thousand PLN. The profit was then equal to $300 - 250 = 50$ thousand PLN. After the change, the revenue reached the value $P_2 = C_2{\cdot}S_2 = 5.4 \cdot 56 = 302.4$ thousand PLN, and the cost $K_2 = 100 + 3S_2 =$ $= 100 + 3 \cdot 56 = 268$ thousand PLN. Thus the profit is now equal to $302.4 - 268 = 34.4$ thousand PLN.

Example 6

The amount of company's customers in the year t ($t = 0$ for year 2005) is described with sequence (k_t). It is known that on average every second customer attracts a new customer the following year. On the other hand, an average of 20% of the customers gives up after one year and 12% after two years. In 2005, the number of customers amounted to 110 thousand, in 2006 – 121 thousand. Find the formula of (k_t) and the expected number of customers in 2017.

Solution

From the above information it follows that the number of customers in year t equals to the number of customers in year ($t – 1$) increased by 50% of the amount of customers in year ($t – 1$) (new customers), decreased by 20% of the amount of customers in year ($t – 1$) (the customers that gave up after one year) and decreased by 12% of the customers from the year ($t – 2$) (customers that gave up after two years). These conditions can be written as:

$$k_t = k_{t-1} + 0.5k_{t-1} - 0.2k_{t-1} - 0.12k_{t-2}.$$

It leads us to the difference equation:

$$k_t - 1.3k + 0.12k_{t-2} = 0 .$$

The characteristic equation has the form:

$$x^2 - 1.3x + 0.12 = 0 .$$

It has two solutions: $x_1 = 0.1$ and $x_2 = 1.2$. It follows that the sequence k_t is described with the formula:

$$k_t = C_1 \cdot 0.1^t + C_2 \cdot 1.2^t .$$

By substituting the initial conditions ($k_0 = 110$, $k_1 = 121$), we can find the constants: $C_1 = 10$, $C_2 = 100$. Finally the sequence has the form:

$$k_t = 10 \cdot 0.1^t + 100 \cdot 1.2^t .$$

Thus the expected amount of customers in 2017 is equal to:

$$k_{12} = 10 \cdot 0.1^{12} + 100 \cdot 1.2^{12} \approx 891.61 \text{ thousand.}$$

EXERCISES

1. Write the relevant differential equation and find the function having:
   a) constant marginal value b;
   b) constant elasticity b.
2. Find the saturation levels of the functions:
   a) $y = \dfrac{130x}{x+15}$;
   b) $y = \dfrac{220x}{x+7}$;
   c) $y = \dfrac{130(x-7)}{x+15}$;
   d) $y = \dfrac{220(x-11)}{x+7}$.
3. The company has established the following relationship between the demand for sausages, measured in terms of sales S (tons per week), and the average income of consumers D (in PLN for 1 person per month) and the price of sausages C: $S = 30\, D^{0.5}\, C^{-1.5}$. Also the relationship between the weekly production cost K (thousand PLN) and production volume Q (in tons per week) has been estimated: $K = 40 + 10Q$. We assume that the weekly production is adapted to the volume of sales. Current sales $S_1 = 12$ tons, price $C_1 = 15$ PLN/kg.
   a) Calculate the approximate change of demand for sausages, if its price increases by 8%.
   b) Determine how in this case will change weekly revenue, cost and profit.
   c) Specify whether the decision taken is right and why.
4. The company has established the following relationship between the demand for washing powder, measured in terms of sales S (tons per week), and the average income of consumers D (in PLN for 1 person per month) and the price of powder C: $S = 120\, D^{0.6}\, C^{-1.0}$. Also the relationship between the weekly production cost K (thousand PLN) and production volume Q (in tons per week) has been estimated: $K = 100 + 3Q$. Current price is $C_1 = 5$ PLN/kg, current sales $Q_1 = 100$ ton.
   a) Calculate the approximate change of demand for powder, if its price increases by 5%.
   b) Determine how in this case will change weekly revenue, cost and profit.
   c) Specify whether the decision taken is right and why.
5. Demand for good D (in thousand pieces) depends on its price P (PLN) and advertising expenses R (thousand PLN). It is known that 1%

increase of price is followed by 0.5% decrease of demand, while 1% increase of advertising causes 2% increase of demand. When the price is 25 PLN and the advertising expenses are 10 thousand PLN, the demand equals to 200 thousand pieces. What should be the price if the advertising expenses are 100 thousand PLN and we want the demand to be 400 thousand pieces?

   a) Find the demand function.
   b) What should be the advertising expenses if the price is 100 PLN and we want the demand to be 400 thousand pieces?

6. The amount of company's customers in the year t ($t = 0$ for year 2010) is described with sequence (a_t). On average every fourth customer attracts a new customer the following year. Find the formula of (a_t) and the expected number of customers in 2017, if:
   a) 5% of the customers give up after one year, 11% after two years, the number of customers in 2010 was 60 thousand, and in 2011 – 56 thousand;
   b) 10% of the customers give up after one year, 5.5% after two years, the number of customers in 2010 was 60 thousand and in 2011 – 13.5 thousand.

SOLUTIONS

1. a) equation: $\frac{dy}{dx} = b$, function: $y = bx + C$ (linear);
   b) equation: $\frac{x}{y} \cdot \frac{dy}{dx} = b$, function: $f(x) = Ce^{bx}$ (power).
2. a) 130; b) 220; c) 130; d) 220.
3. a) the demand will decrease by about 12% (to 10.56 tons);
   b) P_1 = 180,000 PLN, K_1 = 160,000 PLN, Z_1 = 20,000 PLN,
   P_2 = 171,072 PLN, K_2 = 145,600 PLN, Z_2 = 25,472 PLN;
   c) the decision was right – the profit increased.
4. a) the demand will decrease by about 5% (to 95 tons);
   b) P_1 = 500,000 PLN, K_1 = 400,000 PLN, Z_1 = 100,000 PLN,
   P_2 = 498,750 PLN, K_2 = 385,000 PLN, Z_2 = 113,750 PLN;
   c) the decision was right – the profit increased.
5. a) $D(P, R) = 200(\mathrm{P}/25)^{-0.5}(R/10)^2$; b) $R = 20$.
6. a) equation: $a_t - 1.2a_{t-1} + 0.11a_{t-2} = 0$,
   sequence: $a_t = 10 \cdot 0.1^t + 50 \cdot 1.1^t$,
   amount of customers in the year 2017: a_7 = 97.44 thousand;
   b) equation: $a_t - 1.15a_{t-1} + 0.055a_{t-2} = 0$,
   sequence: $a_t = 50 \cdot 0.05^t + 10 \cdot 1.1^t$,
   amount of customers in the year 2017: a_7 = 19.5 thousand.

CHAPTER 13
FINANCIAL MATHEMATICS

13.1. COMPOUND INTEREST, STRAMS OF MONEY AND IRR

THEORY IN A NUTSHELL

The value of money changes over time. Whether you will invest your capital in the bank, or leave in your pocket and wait, its value will fall due to inflation: 100 PLN today is worth more than 100 PLN in one year.
The present value of capital will be denoted by PV, the future value by FV, the number of periods by n, and by i the interest rate per one period (i.e. the percentage by which the value of capital increases as the period, which can be, for example, interest rate on deposit, but also the rate of return on investment or loan rate). The relationship between the current and future value describes the formula:

$$FV = (1+i)^n PV .$$

It can be transformed to the form:

$$PV = \frac{FV}{(1+i)^n} .$$

Compounding is increasing the principal by the interest (after compounding new interest is calculated based upon the principal and the previously calculated interest). The more frequent is the compounding, the higher is the **effective interest rate**. In the above formulae the number of periods defines also the number of times the compounding took place.
If we sum up several payments, we will obtain the so-called **stream of payments**. The future value of a stream (i.e. the **future value of an annuity** if the payments are equal) is defined by the formula:

$$FV = (1+i)^n PV_1 + (1+i)^{n-1} PV_2 + \ldots + (1+i)PV_n = \sum_{k=1}^{n} (1+i)^{n-k+1} PV_k .$$

Here we assume that the payments are made at the beginning of each period (i.e. from above), and the value of the stream is calculated in the end of the last period.

If we want to calculate the present value of future payments made in the end of n periods, i.e. from below (the **present value of an annuity** if the payments are equal), we use the following formula:

$$PV = \frac{FV_1}{(1+i)} + \frac{FV_2}{(1+i)^2} + \ldots + \frac{FV_n}{(1+i)^n} = \sum_{k=1}^{n} \frac{FV_k}{(1+i)^k} .$$

If we want to calculate the present value of infinitely many future payments made in the end of periods (the **present value of a perpetuity** if the payments are equal), we use the following formula:

$$PV = \frac{FV_1}{(1+i)} + \frac{FV_2}{(1+i)^2} + \ldots + \frac{FV_n}{(1+i)^n} + \ldots = \sum_{k=1}^{\infty} \frac{FV_k}{(1+i)^k} .$$

If we know both the future payments and the present value, we can calculate the risk on the investment. The value that we want to find is the **internal rate of return (IRR)**. We find it by solving the following equation with respect to i:

$$PV = \frac{FV_1}{(1+i)} + \frac{FV_2}{(1+i)^2} + \ldots + \frac{FV_n}{(1+i)^n} = \sum_{k=1}^{n} \frac{FV_k}{(1+i)^k} .$$

EXAMPLES

Example 1
We assume that the principal is PV and it increases by $i/100\%$ in each period. Find the formula for the value of capital V_n after n periods of compounding.

Solution
It follows that $V_n = (1 + i)V_{n-1}$, or $V_n - (1 + i)V_{n-1} = 0$. The solution of this difference equation is $V_n = C\,(1 + i)^n$. Since $V_0 = PV$, the constant equals to $C = PV$, hence $V_n = PV\,(1 + i)^n$. This way we proved the formula for the future value.

Example 2
We make a deposit of 1,000 PLN. The interest rate is 40% subject to quarterly compounding. What will be value of deposit after 1.5 years?

Solution
The number of quarters in 1.5 years is $n = 6$. The quarterly interest rate is one fourth of the annual one: $i = 0.4/4 = 0.1$. The present value is $PV = 1{,}000$. It follows that the future value of the deposit is:

$$FV = (1+0.1)^6 \cdot 1{,}000 = 1{,}771.56\,.$$

After 1.5 years we will be able to withdraw 1,771.56 PLN.

Example 3
We want to be able to withdraw 2,000 PLN in two years. The interest rate is 10% subject to semi-annual compounding. How much must we invest?

Solution
The number of semesters in two years is $n = 4$. The semi-annual interest rate is $i = 0.05$, and the future value of deposit is supposed to be $FV = 2{,}000$. It follows that:

$$PV = \frac{2{,}000}{(1+0.05)^4} = 1{,}645.40\,.$$

It means that we have to invest 1,645.40 PLN.

Example 4
We invest today 1,000 PLN and 200 PLN less each quarter (payments from above). The interest rate is 20% yearly subject to quarterly compounding. How much will we collect till the end of year?

Solution
The respective stream of payments has been presented in Fig. 13.1.

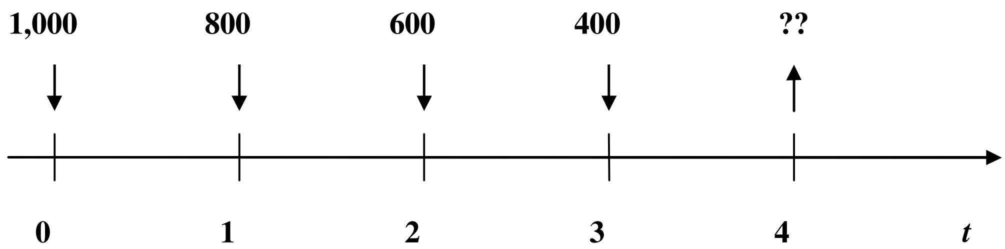

Fig. 13.1. Stream of payments

Quarterly interest rate is $i = 0.05$, the number of quarters is $n = 4$, and the payments in the following quarters are respectively $PV_1 = 1{,}000$, $PV_2 = 800$, $PV_3 = 600$ and $PV_4 = 400$. It follows that the future value of the payments equals to:

$$FV = (1+0.05)^4 \cdot 1{,}000 + (1+0.05)^3 \cdot 800 + (1+0.05)^2 \cdot 600 + (1+0.05)^1 \cdot 400 = \\ = 3{,}223.11.$$

Thus we will collect this way 3,223.11 PLN till the end of year.

Example 5

The investment will bring 1,000 PLN in one year and 2,000 PLN in two years. What is the maximum reasonable value of investment, if the risk-free interest rate equals to 5%?

Solution

We have to find the present value of the investment. We will use the risk–free rate to estimate, how much we would be able to earn without risk. The respective stream of payments has been presented in Fig. 13.2.

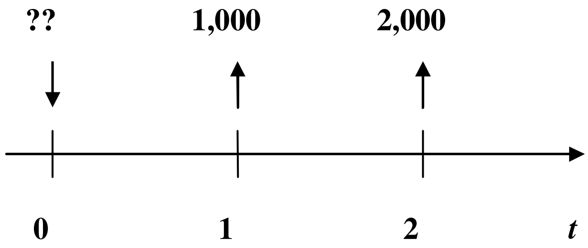

Fig. 13.2. Stream of payments

The interest rate is equal to $i = 0.05$, the number of years is $n = 2$, and the future payments are $FV_1 = 1{,}000$ and $FV_2 = 2{,}000$. By using the appropriate formula we obtain:

$$PV = \frac{1{,}000}{(1+0.05)} + \frac{2{,}000}{(1+0.05)^2} = 2{,}766.44\,.$$

Thus we can earn the given amounts of money without risk, investing today 2,766.44 PLN. It means that 2,766.44 PLN is the highest amount that we should invest.

Example 6

What is the value of deposit that should be made today, if the annual interest rate is 5% and we want to withdraw indefinitely 20,000 PLN in the end of each year?